Journal of Statistical Physics, Vol. 89, Nos. 1/2, 1997

Finite-Dimensional Representations of a Shock Algebra

Eugene R. Speer[1]

Received May 21, 1997

The algebra describing a shock measure in the asymmetric simple exclusion model, seen from a second class particle, has finite-dimensional representations if and only if the asymmetry parameter p of the model and the left and right asymptotic densities $\rho_\pm$ of the shock satisfy $[(1-p)/p]^r = \rho_-(1-\rho_+)/\rho_+(1-\rho_-)$ for some integer $r \geqslant 1$; the minimal dimension of the representation is then $2r$. These representations can be used to calculate correlation functions in the model.

KEY WORDS: Asymmetric simple exclusion process; weakly asymmetric limit; shock profiles; second-class particles; Burgers equation.

1. INTRODUCTION

In ref. 1 the measure describing a shock in the asymmetric simple exclusion model with asymmetry parameter p $(1/2 < p \leqslant 1)$, seen from a second class particle and having left and right asymptotic densities ρ_- and ρ_+ $(0 \leqslant \rho_- < \rho_+ \leqslant 1)$, is derived using a variant of the "matrix method". In this note we study finite dimensional representations of the relevant algebraic structure: specifically, we seek linear operators D, E, and A on a finite dimensional vector space V, and vectors v in V and w in the dual space to V, such that

$$DE - xED = (1-x)[(1-\rho_+)(1-\rho_-)D + \rho_+\rho_- E]; \tag{1a}$$

$$\begin{aligned} DA - xAD &= (1-x)\rho_+\rho_- A, \\ AE - xEA &= (1-x)(1-\rho_+)(1-\rho_-)A; \end{aligned} \tag{1b}$$

[1] Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903.

0022-4715/97/1000-0169$12.50/0

$$(D+E)v = v, \qquad (D+E)^* w = w; \tag{1c}$$

$$(w, Av) = 1; \tag{1d}$$

where $x = (1-p)/p$. For the moment we take $x > 0$ $(p < 1)$, $\rho_- > 0$, and $\rho_+ < 1$; the remaining special cases will be discussed briefly later. We will show that finite dimensional representations exist if and only if

$$x^r = \frac{\rho_-(1-\rho_+)}{\rho_+(1-\rho_-)} \tag{2}$$

for some positive integer r, and that the minimal dimension of the representation is then $2r$.

2. CONDITIONS FOR EXISTENCE OF FINITE DIMENSIONAL REPRESENTATIONS

Suppose that V is a finite dimensional vector space and that the operators D, E, and A and vectors v and w satisfy (1). For convenience we introduce the constants $a = \rho_+(1-\rho_-)$ and $b = \rho_-(1-\rho_+)$ and the operators $\hat{D} = D - \rho_-\rho_+$ and $\hat{E} = E - (1-\rho_-)(1-\rho_+)$, which satisfy

$$\hat{D}\hat{E} - x\hat{E}\hat{D} = (1-x)\,ab. \tag{3}$$

Our starting point is the treatment of (1a) in ref. 2 (finite dimensional representations of (1a) were also studied in ref. 3), where it is observed that since $\hat{D}\hat{E}$ and $\hat{E}\hat{D}$ have the same spectrum, (3) implies that if λ is a point of this spectrum, then so is $ab + x(\lambda - ab)$. This in turn implies that the spectrum is the single point ab, so that $\hat{D}$ is invertible. The operator

$$P = \hat{E} - ab\hat{D}^{-1},$$

was then used in ref. 2 to study finite dimensional representations of (1a). Here, where we must consider the operator A as well as D and E, these considerations lead us to observe that (1) may be replaced by

$$\hat{D}P = xP\hat{D}, \qquad \hat{D}A = xA\hat{D}, \qquad AP = xPA; \tag{4a}$$

$$(G+P)v = v, \qquad (G+P)^* w = w; \tag{4b}$$

$$(w, Av) = 1; \tag{4c}$$

where $G = \hat{D} + ab\hat{D}^{-1} - (a+b)I$. (Equation (4a) describes the quadratic algebra $A_x^{3|0}$.[4])

The space V may be decomposed as a direct sum

$$V = \bigoplus_{\gamma \in \Sigma} V_\gamma, \tag{5}$$

where Σ is the spectrum of $\hat{D}$ and, for $\gamma \in \Sigma$, V_γ is the $\hat{D}$-invariant subspace naturally associated with this eigenvalue: $V_\gamma = \{v \in V: (\hat{D} - \gamma I)^k v = 0$ for some $k \geq 1\}$ (see, e.g., ref. 5). We choose an inner product in V in such a way that (5) is an orthogonal sum, and for $u \in V$ let u_γ denote the orthogonal projection of u on V_γ. The dual space to V may now be identified with V itself. Note that (4a) implies that P and A map V_γ to $V_{\gamma x}$ (with P and A vanishing on V_γ if $x\gamma \notin \Sigma$) and that P^* and A^* map V_γ to $V_{\gamma/x}$.

Since the decomposition (5) reduces the operator G, (4b) implies that

$$Gv_\gamma = -Pv_{\gamma/x} \qquad \text{and} \qquad G^* w_\gamma = -P^* w_{\gamma x} \tag{6}$$

(with the convention that $v_\gamma = 0$ if $\gamma \notin \Sigma$). It follows that if $v_\gamma \neq 0$ then either $v_{\gamma/x} \neq 0$ or $Gv_\gamma = 0$; because the unique eigenvalue of G in V_γ is $\gamma + ab/\gamma - a - b$, the latter is possible only if $\gamma = a$ or $\gamma = b$. We conclude that v_γ can be nonzero only if $\gamma = ax^k$ or $\gamma = bx^k$ for some $k \geq 0$. Similarly, w_γ can be nonzero only if $\gamma = b/x^j$ or $\gamma = a/x^j$ for some $j \geq 0$. Because (5) is an orthogonal sum, (4c) implies that $ax^{k+1} = b/x^j$ for some k and j, i.e. (2) must hold for $r = j + k + 1$.

We will prove that the dimension of V is at least $2r$ by showing that the vectors

$$v_a, v_{ax}, \dots, v_{ax^{r-1}}, w_{b/x^{r-1}}, \dots, w_{b/x}, w_b \tag{7}$$

are nonzero and pairwise orthogonal. It is convenient to introduce a functional calculus for a restricted set of functions: for an invertible operator O on V and a function of the form $f(z) = z^{-n} q(z)$, with n a nonnegative integer and q a polynomial, $f(O) = O^{-n} q(O)$. The mapping $f \mapsto f(O)$ is linear and multiplicative, and $f(O) = 0$ if f has a zero of sufficiently high order at all eigenvalues of O.

From (4a) it follows that $f(\hat{D}) A = A f(x\hat{D})$ and $f(\hat{D}) P = P f(x\hat{D})$ for any f and hence, again using (4a), that

$$A f(\hat{D}) P = x P f(\hat{D}) A. \tag{8}$$

The operator G defined above is $g(\hat{D})$, with $g(z) = z + abz^{-1} - (a+b)$. Let $h(z)$ be a polynomial vanishing to high order at $z = a$ and $z = b$, and agreeing with $-1/g(z)$ to high order at other points of Σ, and let $H = h(\hat{D})$, so that $GH = HG$ acts as the negative of the identity on V_γ if $\gamma \neq a, b$. From (6), $v_{ax^k} = HPv_{ax^{k-1}}$ and $w_{b/x^k} = H^*P^*w_{b/x^{k-1}}$ for $k = 1,..., r-1$. Then if $0 \leqslant k \leqslant r-2$, from (8),

$$\begin{aligned}(w_{b/x^{r-k-1}}, Av_{ax^k}) &= (H^*P^*w_{b/x^{r-k-2}}, Av_{ax^k})\\ &= (w_{b/x^{r-k-2}}, PHAv_{ax^k})\\ &= x^{-1}(w_{b/x^{r-k-2}}, AHPv_{ax^k})\\ &= x^{-1}(w_{b/x^{r-k-2}}, Av_{ax^{k+1}}).\end{aligned} \tag{9}$$

Now (4c) implies that $(W_{b/x^{r-k-1}}, Av_{ax^k}) \neq 0$ for some k with $0 \leqslant k \leqslant r-1$, and then (9) implies that this holds for all such k, and hence all the vectors in (7) are nonzero.

Finally, we claim that

$$(f_1(\hat{D})^* w_{b/x^{r-k}}, f_2(\hat{D})\, v_{ax^k}) = 0 \tag{10}$$

for any functions f_1, f_2 and any k, $1 \leqslant k \leqslant r-1$; (10) for $f_1 = f_2 = 1$, together with the orthogonality of V_{ax^k} and V_{ax^j} for $k \neq j$, implies that the vectors of (7) are pairwise orthogonal. For $k = 1,..., r-1$,

$$\begin{aligned}&(f_1(\hat{D})^* w_{b/x^{r-k}}, f_2(\hat{D})\, v_{ax^k})\\ &\quad= (f_1(\hat{D})^* w_{b/x^{r-k}}, f_2(\hat{D})\, HPv_{ax^{k-1}})\\ &\quad= (f_1(x\hat{D})^* w_{b/x^{r-k}}, f_2(x\hat{D})\, h(x\hat{D})\, v_{ax^{k-1}})\\ &\quad= -(f_1(x\hat{D})^* G^*w_{b/x^{r-k+1}}, f_2(x\hat{D})\, h(x\hat{D})\, v_{ax^{k-1}}).\end{aligned} \tag{11}$$

For $k = 1$, (11) vanishes because $Gv_a = 0$ by (6); for $k > 1$ it vanishes by induction. This verifies the claim and completes the proof that the elements of (7) are linearly independent.

3. REPRESENTATIONS OF MINIMAL DEGREE

The above considerations lead, for any r, to the construction of a representation with the minimal dimension $2r$. In the representation we construct the operator D and hence G and H will be diagonal, so that from (6), v_{ax^k} is proportional to $P^k v_a$ and w_{b/x^k} to $(P^*)^k w_b$. We normalize the inner product so that the vectors $P^k v_a$ and $(P^*)^k w_b$, $k = 0,..., r-1$, form an orthonormal basis. Taking these as the standard basis in $\mathbb{R}^{2r}$, we may represent the operators as matrices, using a block decomposition into $r \times r$ blocks:

$$\hat{D} = \begin{bmatrix} \hat{D}^{11} & 0 \\ 0 & \hat{D}^{22} \end{bmatrix}, \quad P = \begin{bmatrix} P^{11} & 0 \\ 0 & P^{22} \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 \\ A^{21} & 0 \end{bmatrix}, \tag{12}$$

with

$$\hat{D}^{11} = \begin{bmatrix} a & & & \\ & ax & & \\ & & \ddots & \\ & & & ax^{r-1} \end{bmatrix}, \quad \hat{D}^{22} = \begin{bmatrix} ax & & & \\ & ax^2 & & \\ & & \ddots & \\ & & & b \end{bmatrix},$$

$$P^{11} = P^{22} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix},$$

and

$$A^{21} = \begin{bmatrix} 1 & & & \\ & x & & \\ & & \ddots & \\ & & & x^{r-1} \end{bmatrix}.$$

Finally, $w = [0 \;\; w^2]$ and $v = [{}^{v^1}_{0}]$, with

$$w^2 = Z^{-1}[\kappa_{r-1} \;\; \cdots \;\; \kappa_1 \;\; \kappa_0], \qquad v^1 = Z^{-1} \begin{bmatrix} \kappa_0 \\ \kappa_1 \\ \vdots \\ \kappa_{r-1} \end{bmatrix},$$

where $\kappa_k = \prod_{i=1}^{k} g(ax^i)^{-1}$ (with $\kappa_0 = 1$) and $Z = [\kappa_{r-1}(1-x^r)/(1-x)]^{1/2}$.

4. THE TOTALLY ASYMMETRIC CASE

It can be verified by the methods above that when $x = 0$ or $b = 0$ ($\rho_- = 0$ or $\rho_+ = 1$), finite dimensional representations exist if and only if $x = b = 0$. The representation of minimal dimension is the two dimensional representation found above and in ref. 1.

5. ASYMPTOTIC BEHAVIOR OF THE PROFILE

The rate of decay of correlations in the shock measure is governed by the eigenvalues of $D+E=I-G$ other than 1; for simplicity we discuss only the decay of $\langle \tau_n \rangle = (w, A(D+E)^{n-1} Dv)$, the density n sites in front of the second class particle, to its asymptotic value ρ_+. The eigenvalues of $I-G$ are 1 and

$$\lambda_k = 1 - g(ax^k) = 1 - a - b + ax^k + b/x^k, \; k = 1,..., r-1.$$

Since $\lambda_k = \lambda_{r-k}$, each of these eigenvalues, other than 1 and $\lambda_{r/2}$ for r even, is doubly degenerate in each of the two $r \times r$ blocks in (12). The explicit formulae above for $\hat{D}$ and P show that there is only one eigenvector in each block, however, so that

$$\langle \tau_n \rangle = \rho_+ + \sum_{1 \leq k \leq r/2} [\alpha_k + (n-1)\beta_k] \lambda_k^{n-1},$$

for constants α_k, β_k with $\beta_{r/2} = 0$ when r is even. We discuss in more detail the cases $r = 1$, 2, and 3:

$r = 1$; $x = b/a$. The finite dimensional representation in this case is discussed in ref. 1. The shock measure is Bernoulli, with no correlations, and $\langle \tau_n \rangle = \rho_+$ for all n.

$r = 2$; $x^2 = b/a$. It was shown in ref. 1 that the model exhibits distinct forms of leading asymptotic behavior in the two regions $x^2 > b/a$ and $x^2 < b/a$; the behavior on the surface separating these regions was not discussed. This is precisely the surface on which the four dimensional representation given above applies. It yields the exact result

$$\langle \tau_n \rangle = \rho_+ - \frac{x(1-x)}{1+x} a\lambda_1^{n-1}.$$

$r = 3$; $x^3 = b/a$. In this case the asymptotic formula of ref. 1 is exact: for all n.

$$\langle \tau_n \rangle = \rho_+ - \frac{x^2(1-x)^3(1+x)}{1+x+x^2} a^2(n-1)\lambda_1^{n-2} - x(1-x) a\lambda_1^{n-1}.$$

REFERENCES

1. B. Derrida, J. L. Lebowitz, and E. R. Speer, Shock profiles for the partially asymmetric simple exclusion process, *J. Stat. Phys.* **89**:135–167 (1997), and references therein.

2. K. Mallick and S. Sandow, Finite dimensional representations of the quadratic algebra: applications to the exclusion process, *J. Phys. A* **30**:4513–4526 (1997).
3. F. H. L. Essler and V. Rittenberg, Representation of the quadratic algebra and partially asymmetric diffusion with open boundaries, *J. Phys. A* **29**:3375–3407 (1996).
4. C. Kassel, *Quantum Groups* (Springer-Verlag, New York, 1995).
5. P. R. Halmos, *Finite Dimensional Vector Spaces* (van Nostrand, Princeton, 1958).